

ON THE OPERATOR-VALUED μ -COSINE FUNCTIONS

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ABSTRACT. Let $(G, +)$ be a topological abelian group with a neutral element e and let $\mu : G \rightarrow \mathbb{C}$ be a continuous character of G . Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and let $\mathbf{B}(\mathcal{H})$ be the algebra of all linear continuous operators of \mathcal{H} into itself. A continuous mapping $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ will be called an operator-valued μ -cosine function if it satisfies both the μ -cosine equation

$$\Phi(x+y) + \mu(y)\Phi(x-y) = 2\Phi(x)\Phi(y), \quad x, y \in G$$

and the condition $\Phi(e) = I$, where I is the identity of $\mathbf{B}(\mathcal{H})$. We show that any hermitian operator-valued μ -cosine functions has the form

$$\Phi(x) = \frac{\Gamma(x) + \mu(x)\Gamma(-x)}{2}$$

where $\Gamma : G \rightarrow \mathbf{B}(\mathcal{H})$ is a continuous multiplicative operator. As an application, positive definite kernel theory and W. Chojnacki's results on the uniformly bounded normal cosine operator are used to give explicit formula of solution of the cosine equation.

1. INTRODUCTION

1.1. The μ -cosine equation, also called the pre-d'Alembert equation, on abelian group G is the equation

$$(1.1) \quad f(x+y) + \mu(y)f(x-y) = 2f(x)f(y), \quad x, y \in G$$

where $f : G \rightarrow \mathbb{C}$ is the unknown. Davison [8] gave solution of (1.1) in terms of traces of certain representations of G on \mathbb{C}^2 . In [22] Stetkær proves that a non-zero solution of (1.1) has the form

$$(1.2) \quad f(x) = \frac{\chi(x) + \mu(x)\chi(-x)}{2}, \quad x \in G,$$

where χ is a character of G . In the case where $\mu = 1$, equation (1.1) becomes the classic cosine functional equation (also called the d'Alembert functional equation)

$$(1.3) \quad f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G.$$

Several mathematicians studied the equation (1.3). The monographs by Aczél [2] and by Aczél and Dhombres [3] have references and detailed discussions. The main purpose of this work is to extend equation (1.1) to functions taking values in the algebra $\mathbf{B}(\mathcal{H})$ of bounded operators on a Hilbert space \mathcal{H} .

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1.2. Throughout this paper, G will be a topological abelian group with the unit element e . The space of continuous complex-valued functions is denoted by $\mathcal{C}(G)$ and the set of all continuous homomorphisms $\gamma : G \rightarrow \mathbb{C} \setminus \{0\}$ by $\mathcal{M}(G)$. Let $\mu : G \rightarrow \mathbb{C}^*$ be a continuous character of the group G i.e. $\mu \in \mathcal{M}(G)$ such that $\mu(e) = 1$. For all $f \in \mathcal{C}(G)$ we define the function \check{f} by $\check{f}(x) = f(-x)$. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over \mathbb{C} and let $\mathbf{B}(\mathcal{H})$ be the algebra of all linear continuous operators of \mathcal{H} into itself with the usual operator norm denoted $\|\cdot\|$. A mapping $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ is said to be hermitian if it satisfies $\Phi^*(x) = \Phi(-x)$ for all $x \in G$, where $\Phi^*(x)$ is the adjoint operator of $\Phi(x)$. A continuous mapping $\Gamma : G \rightarrow \mathbf{B}(\mathcal{H})$ is said to be a multiplicative operator if $\Gamma(x+y) = \Gamma(x)\Gamma(y)$ for all $x, y \in G$ and $\Gamma(e) = I$. Also we say that a continuous mapping $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ is an operator valued μ -cosine function if it satisfies both the μ -cosine functional equation

$$(1.4) \quad \Phi(x+y) + \mu(y)\Phi(x-y) = 2\Phi(x)\Phi(y), \quad x, y \in G$$

and the conditions $\Phi(e) = 1$. The scalar case of (1.4) is given by the equation (1.1). For $\mu = 1$ we obtain the cosine functional equation

$$(1.5) \quad \Phi(x+y) + \Phi(x-y) = 2\Phi(x)\Phi(y), \quad x, y \in G.$$

Several variants of (1.5) has been studied by Kiszyński [10] and [11], Székelyhidi [23], Chojnacki [6] and [7], Stetkær [19], [20] and [21].

1.3. The main purpose of this work is to solve the equation (1.4), where the unknown Φ is an hermitian continuous functions on G taking its values in $\mathbf{B}(\mathcal{H})$ or in the algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices. By using positive definite kernels and linear algebra theory we find that any hermitian continuous solution of (1.4) has the form $\Phi(x) = \frac{\Gamma(x) + \mu(x)\Gamma(-x)}{2}$ where $\Gamma : G \rightarrow \mathbf{B}(\mathcal{H})$ is a continuous multiplicative operator.

1.4. Notation and preliminary.

Definition 1.1. A continuous function $K : G \times G \rightarrow \mathbb{C}$ is said to be a positive definite kernel on G if for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in G$ and arbitrary complex numbers c_1, c_2, \dots, c_n we have

$$(1.6) \quad \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} K(x_i, x_j) \geq 0.$$

We provide some known results on positive definite kernel theory. For more details we refer to [15].

Proposition 1.2. Let K be a positive definite kernel on G and let

$$V_K := \text{span}\{K(x, \cdot) : x \in G\}.$$

Then

i) $V_K \subset \mathcal{C}(G)$,

ii) V_K is equipped with the inner product

$$\langle f, g \rangle_K = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\beta_j} K(x_i, x_j)$$

where

$$f = \sum_{i=1}^n \alpha_i K(x_i, \cdot), \quad g = \sum_{j=1}^m \beta_j K(x_j, \cdot),$$

$x_1, \dots, x_{\sup(m,n)} \in G$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{C}$.

Let \mathcal{H}_K be the completion of V_K . Then $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$ is a Hilbert space of continuous functions on G . The function K is the reproducing kernel of the Hilbert space \mathcal{H}_K .

Theorem 1.3. *Let K be a positive definite kernel on G . Then there exists a Hilbert space $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$ and a continuous mapping*

$$T : G \longrightarrow \mathcal{H}_K, x \longmapsto K(x, \cdot),$$

such that

1) $K(x, y) = \langle T(x), T(y) \rangle_K$ for all $x, y \in G$.

2) $\text{span}\{T(x) : x \in G\}$ is dense in \mathcal{H}_K .

Moreover, the pair (\mathcal{H}_K, T) is unique in the following way : if another pair (\mathcal{L}, U) satisfies (1) and (2), there exists a unique unitary isomorphism $\Psi : \mathcal{H}_K \longrightarrow \mathcal{L}$ such that $U = \Psi \circ T$.

For all $f \in \mathcal{C}(G)$ and for all $x, y \in G$ we define

$$(1.7) \quad K_f(x, y) := \frac{1}{2} \{f(-y + x) + \mu(x)f(-y - x)\}.$$

and

$$(1.8) \quad f(x) = f_\mu^+(x) + f_\mu^-(x), \quad x \in G,$$

where $f_\mu^+(x) = \frac{f(x) + \mu(x)f(-x)}{2}$ and $f_\mu^-(x) = \frac{f(x) - \mu(x)f(-x)}{2}$

2. GENERAL PROPERTIES

Proposition 2.1. *Let $\Phi : G \longrightarrow \mathbf{B}(\mathcal{H})$ be a solution of (1.4). Then*

i) $\Phi(-x) = \mu(-x)\Phi(x)$ for all $x \in G$.

ii) $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$.

iii) For all invertible operator $S \in \mathbf{B}(\mathcal{H})$ we have $S\Phi(x)S^{-1}$ for all $x \in G$ is a solution of (1.4).

Proof. i) For all $x, y \in G$ we have

$$\begin{aligned} 2\Phi(x)\Phi(y) &= \Phi(x+y) + \mu(y)\Phi(x-y) \\ &= \mu(y)(\Phi(x-y) + \mu(-y)\Phi(x+y)) \\ &= 2\mu(y)\Phi(x)\Phi(-y). \end{aligned}$$

From which we get that

$$(2.1) \quad \Phi(x)\Phi(y) = \mu(y)\Phi(x)\Phi(-y).$$

Setting $x = e$ in (2.1) and $\Phi(e) = I$ we get that $\Phi(y) = \mu(y)\Phi(-y)$ for all $y \in G$.

ii) For all $x, y \in G$ we have

$$\begin{aligned} 2\Phi(y)\Phi(x) &= \Phi(y+x) + \mu(x)\Phi(y-x) \\ &= \Phi(x+y) + \mu(x)\mu(y-x)\Phi(x-y) \\ &= \Phi(x+y) + \mu(y)\Phi(x-y) \\ &= 2\Phi(x)\Phi(y). \end{aligned}$$

From which we get that $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$.

iii) For all $x, y \in G$ we have

$$\begin{aligned} S\Phi(x+y)S^{-1} + \mu(y)S\Phi(x-y)S^{-1} &= S(\Phi(x+y) + \mu(y)\Phi(x-y))S^{-1} \\ &= S2\Phi(x)\Phi(y)S^{-1} \\ &= 2S\Phi(x)S^{-1}S\Phi(y)S^{-1}. \end{aligned}$$

From which we get that

$$S\Phi(x+y)S^{-1} + \mu(y)S\Phi(x-y)S^{-1} = 2S\Phi(x)S^{-1}S\Phi(y)S^{-1}$$

for all $x, y \in G$. Furthermore we have

$$S\Phi(e)S^{-1} = I.$$

□

Proposition 2.2. *Let $M : G \longrightarrow \mathbf{B}(\mathcal{H})$ be a multiplicative operator. Then*

$$\Phi(x) = \frac{M(x) + \mu(x)M(-x)}{2}, \quad x \in G$$

is an operator-valued μ -cosine functions.

Proof. Since $M(x+y) = M(x)M(y)$ for all $x, y \in G$ and $M(e) = I$, we get by easy computations that Φ is an operator-valued μ -cosine functions. □

By easy computations we get the following proposition

Proposition 2.3. *For all $f \in \mathcal{C}(G)$ we have the following statements*

- i) the mapping $(x, y) \longmapsto K_f(x, y)$ is continuous.*
- ii) $K_f(-x, y) = \mu(-x)K_f(x, y)$ for all $x, y \in G$.*
- iii) $K_f(0, -y) = f(y)$ for all $y \in G$.*
- 4i) $K_f(x, 0) = f_\mu^+(x)$ for all $x \in G$.*

We need the following proposition in the main result

Proposition 2.4. *Let $\Phi : G \longrightarrow \mathbf{B}(\mathcal{H})$ be a solution of (1.4) such that $\Phi(x)^* = \Phi(-x)$ for all $x \in G$ and let $f : G \longrightarrow \mathbb{C}$, $x \longmapsto \langle \Phi(x)\xi, \xi \rangle$ for $\xi \in \mathcal{H}$. Then*

- i) $\overline{f(-x)} = \mu(-x)f(x)$ for all $x \in G$.*
- ii) $\overline{f(x)} = f(-x)$ for all $x \in G$.*
- iii) $K_f(x, y) = \langle \Phi(x)\xi, \Phi(y)\xi \rangle$ for all $x, y \in G$.*
- 4i) K_f is a positive definite kernel.*
- 5i) $K_f(x, \cdot) = \frac{1}{2}\{(R_{-x}\check{f}) + \mu(x)(R_x\check{f})\}$ for all $x \in G$ where R is the right regular representation of G .*

Proof. i) Since $\Phi(-x) = \mu(-x)\Phi(x)$ for all $x \in G$ we get that

$$f(-x) = \langle \Phi(-x)\xi, \xi \rangle = \langle \mu(-x)\Phi(x)\xi, \xi \rangle = \mu(-x)\langle \Phi(x)\xi, \xi \rangle = \mu(-x)f(x).$$

ii) for all $x \in G$ we have

$$\overline{f(x)} = \overline{\langle \Phi(-x)\xi, \xi \rangle} = \langle \xi, \Phi(x)\xi \rangle = \langle \Phi(x)^*\xi, \xi \rangle = \langle \Phi(-x)\xi, \xi \rangle = f(-x).$$

iii) For all $x, y \in G$ we have

$$\begin{aligned} K_f(x, y) &= \frac{1}{2}\{f(-y+x) + \mu(x)f(-y-x)\} \\ &= \frac{1}{2}\{(\langle \Phi(-y+x)\xi, \xi \rangle + \mu(x)\langle \Phi(-y-x)\xi, \xi \rangle)\} \\ &= \langle \Phi(-y)\Phi(x)\xi, \xi \rangle \\ &= \langle \Phi(y)^*\Phi(x)\xi, \xi \rangle \\ &= \langle \Phi(x)\xi, \Phi(y)\xi \rangle \end{aligned}$$

4i) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in G$ and arbitrary complex numbers c_1, c_2, \dots, c_n we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} K_f(x_i, x_j) &= \sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \langle \Phi(x_i) \xi, \Phi(x_j) \xi \rangle \\ &= \left\langle \sum_{i=1}^n c_i \Phi(x_i) \xi, \sum_{j=1}^n c_j \Phi(x_j) \xi \right\rangle \\ &= \left\| \sum_{i=1}^n c_i \Phi(x_i) \right\|^2 \geq 0. \end{aligned}$$

5i) For all $x, y \in G$ we have

$$\begin{aligned} K_f(x, y) &= \frac{1}{2} \{ f(-y + x) + \mu(x) f(-y - x) \} \\ &= \frac{1}{2} \{ (R_x f)(-y) + \mu(x) (R_{-x} f)(-y) \} \\ &= \frac{1}{2} \{ (R_x \check{f})(y) + \mu(x) (R_{-x} \check{f})(y) \} \end{aligned}$$

Since $(R_x \check{f}) = R_{-x} \check{f}$ for all $x \in G$ it follows that $K_f(x, y) = \frac{1}{2} \{ R_{-x} \check{f}(y) + \mu(x) (R_{-x} \check{f})(y) \}$ for all $x, y \in G$. So that we have $K_f(x, \cdot) = \frac{1}{2} \{ (R_{-x} \check{f}) + \mu(x) (R_x \check{f}) \}$ for all $x \in G$. \square

3. MAIN RESULT

In the next theorem we solve the equation (1.4).

Theorem 3.1. *Let $\Phi : G \longrightarrow \mathbf{B}(\mathcal{H})$ be an hermitian operator-valued μ -cosine functions. Then there exists a multiplicative operator $M : G \longrightarrow \mathbf{B}(\mathcal{H})$ such that*

$$\Phi(x) = \frac{M(x) + \mu(x)M(-x)}{2}, \quad x \in G.$$

Proof. Let $\xi \in \mathcal{H}$. By the same way as in the proof of Theorem 2.2 in [1], we can suppose that the vector ξ is cyclic

Let now $\varphi(x) = \langle \Phi(x) \xi, \xi \rangle$ for all $x \in G$. For all $x, y \in G$ we have $K_\varphi(x, y) = \langle \Phi(x) \xi, \Phi(y) \xi \rangle$. So that K_φ is a positive definite kernel. According to Theorem 1.3 there exists a Hilbert space $(\mathcal{H}_\varphi, \langle \cdot, \cdot \rangle_\varphi)$ and a mapping $T : G \longrightarrow \mathcal{H}_\varphi$, $x \longmapsto K_\varphi(\cdot, x)$ such that

$$\begin{aligned} K_\varphi(x, y) &= \langle K_\varphi(x, \cdot), K_\varphi(y, \cdot) \rangle \\ &= \langle \Phi(x) \xi, \Phi(y) \xi \rangle \end{aligned}$$

and a unique unitary isomorphism $\psi : \mathcal{H}_\varphi \longrightarrow \mathcal{H}$ such tat

$$(3.1) \quad \Phi(x) \xi = \psi(K_\varphi(x, \cdot)) = \frac{1}{2} \psi[(R_x \check{\varphi}) + \mu(x)(R_{-x} \check{\varphi})].$$

Since $\Phi(e) = I$ we get by setting $x = e$ in (3.1) that $\xi = \psi(\check{\varphi})$. From which we get that $\check{\varphi} = \psi^{-1}(\xi)$. We show that $(R_x \check{\varphi}) = R_{-x} \check{\varphi}$ and $(R_{-x} \check{\varphi}) = R_x \check{\varphi}$ for all $x \in G$. So that for all $x \in G$ and $\xi \in \mathcal{H}$ we have

$$\Phi(x) \xi = \frac{1}{2} \psi[(R_{-x} + \mu(x)R_x) \psi^{-1}(\xi)].$$

Hence $\Phi(x) = \psi \circ R(x) \circ \psi^{-1}$ for all $x \in G$.

Since $R(x) = \frac{R_{-x} + \mu(x)R_x}{2}$ for all $x \in G$ we get that

$$\Phi(x) = \frac{\psi \circ R_{-x} \circ \psi^{-1} + \mu(x)\psi \circ R_x \circ \psi^{-1}}{2}.$$

Setting $M(x) = \psi \circ R_{-x} \circ \psi^{-1}$ for all $x \in G$. We have for all $x, y \in G$ that

$$\begin{aligned} M(x+y) &= \psi \circ R_{-x-y} \circ \psi^{-1} \\ &= \psi \circ R_{-x} \circ R_{-y} \circ \psi^{-1} \\ &= \psi \circ R_{-x} \circ \psi \circ \psi^{-1} \circ R_{-y} \circ \psi^{-1} \\ &= M(x)M(y). \end{aligned}$$

and that $M(e) = \psi \circ R_e \circ \psi^{-1} = I$.

Finally we have that $\Phi(x) = \frac{M(x) + \mu(x)M(-x)}{2}$ for all $x \in G$ where $M : G \rightarrow \mathcal{H}$ is a multiplicative operator. This ends the proof of theorem. \square

In the next corollary we determine solutions of (1.4) taking their values in the complex $n \times n$ matrices

Corollary 3.2. *Let $\Phi : G \rightarrow M_n(\mathbb{C})$ be a continuous hermitian solution of (1.4). Then there exists $A \in GL(n, \mathbb{C})$ such that*

$$(3.2) \quad \Phi(x) = A \frac{E(x) + \mu(x)E(-x)}{2} A^{-1}, \quad x \in G$$

where $E : G \rightarrow M_n(\mathbb{C})$ has the form

$$\begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \gamma_i & \dots \\ 0 & 0 & \dots & \gamma_n \end{pmatrix}$$

where $\gamma_1, \dots, \gamma_n \in M(G)$ and $\gamma_i \neq \gamma_j$ for all $i, j \in \{1, \dots, n\}$ such that $i \neq j$

Proof. since $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$ and $\Phi(x)^* = \Phi(-x)$ for all $x \in G$ it follows that $\Phi(x)$ for all $x \in G$ can be diagonalized simultaneously. So there exists $A \in GL(n, \mathbb{C})$ such that

$$\Phi(x) = A \begin{pmatrix} \omega_1 & 0 & \dots & 0 \\ 0 & \omega_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \omega_i & \dots \\ 0 & 0 & \dots & \omega_n \end{pmatrix} A^{-1}.$$

Since $\Phi(x+y) + \mu(y)\Phi(x-y) = 2\Phi(x)\Phi(y)$ for all $x, y \in G$ it follows that $\omega_i(x+y) + \mu(y)\omega_i(x-y) = 2\omega_i(x)\omega_i(y)$ for all $i \in \{1, \dots, n\}$. According to [22] there exists $\gamma_i \in \mathcal{M}(G)$ for all $i \in \{1, \dots, n\}$. such that $\omega(x) = \frac{\gamma_i(x) + \mu(x)\gamma_i(-x)}{2}$ for all $x \in G$. So

$$\Phi(x) = A \frac{E(x) + \mu(x)E(-x)}{2} A^{-1} \text{ where } E = \begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \gamma_i & \dots \\ 0 & 0 & \dots & \gamma_n \end{pmatrix} \text{ and } \gamma_i \in \mathcal{M}(G) \text{ such}$$

that $\gamma_i \neq \gamma_j$ for $i \neq j$. This ends the proof of corollary \square

4. APPLICATIONS

Throughout this section we adhere to the terminology used in [7]. Let G be a locally compact commutative group and let $\mu = 1$. A mapping $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ will be said to be uniformly bounded if $\sup\{\|\Phi(x)\| : x \in G\} < +\infty$. The hermitian operator-valued cosine functions is denoted by $*$ -operator-valued cosine functions in [7].

According to Theorem 1 in [7] we get the following proposition

Proposition 4.1. *Let $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ be a uniformly bounded operator-valued cosine functions. Then there is an invertible $S \in \mathbf{B}(\mathcal{H})$ such that $\Psi(x) = S\Phi(x)S^{-1}$ for all $x \in G$ is an hermitian operator-valued cosine functions*

In the next theorem we use our study to solve the equation (1.5)

Theorem 4.2. *Let $\Phi : G \rightarrow \mathbf{B}(\mathcal{H})$ be a uniformly bounded operator-valued cosine functions. Then there is an invertible $S \in \mathbf{B}(\mathcal{H})$ and a multiplicative operator $M : G \rightarrow \mathbf{B}(\mathcal{H})$ such that*

$$\Phi(x) = S \frac{M(x) + M(-x)}{2} S^{-1}, \quad x \in G.$$

Proof. By using Proposition 4.1 we get that $\Psi(x) = S\Phi(x)S^{-1}$ is a solution of (1.5) such that $\Psi(x)^* = \Psi(x^{-1})$ for all $x \in G$. According to Theorem 3.1 we get the remainder. \square

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